Lecture 4

Tuesday, September 7, 2021 10:24 PM

* Prayer
* Spiritual thought: to build a house, one needs to build the foundation first. Every complicated task should start with a simple step.

The simplest kind of differential eqs. is the 1st order linear ODE.

$$a(t)\,y' + b(t)\,y + c(t) = 0$$

$$\rightsquigarrow \quad y' + p(t)\,y = q(t) \qquad (*)$$

How to solve this equation?

Method of integrating factor: multiply both sides of $(*)$ by a factor $\mu(t)$ such that the left hand side is an exact derivative.

$$\underbrace{\mu y' + \mu p y}_{= (\ldots)'} = \mu q$$

$(\ldots) = \mu y$

$\rightsquigarrow$ Want $\mu' = \mu p \rightsquigarrow \dfrac{\mu'}{\mu} = p$

$\rightsquigarrow \ln\mu = \int p$

$\rightsquigarrow \boxed{\mu = e^{\int p}}$

This is an integrating factor.

<u>Algorithm</u> of solving $y' + p(t)y = g(t)$: $(*)$

- Find <u>one</u> antiderivative of p : $P(x) = \int p\,dx$
- Integrating factor $\mu(x) = e^{P(x)}$
- Multiplying both sides of $(*)$ by μ ;

$$(\mu y)' = \mu g \rightsquigarrow \mu y = \int \mu g$$

$$\rightsquigarrow y(x) = \frac{1}{\mu(x)} \int \mu g\,dx$$

<u>Ex</u>: $x^2 y' + xy = 1$, $y(1) = 2$

$$\rightsquigarrow y' + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{\frac{1}{x^2}}_{g(x)} \qquad (*)$$

$$P(x) = \int p(x)\,dx = \ln x$$

Integrating factor: $\mu(x) = e^{P(x)} = e^{\ln x} = x$

$$(*) \rightsquigarrow xy' + y = \frac{1}{x} \rightsquigarrow (xy)' = \frac{1}{x}$$

$$\rightsquigarrow xy = \int \frac{1}{x}\,dx = \ln x + C$$

$$\rightsquigarrow y = \frac{1}{x}(\ln x + C)$$

$$y(1) = 2 \rightsquigarrow C = 2 \rightsquigarrow \boxed{y = \frac{1}{x}(\ln x + 2)}$$

Ex

$$x^2 y' + xy = 1, \quad y(-1) = 2$$

$$\rightsquigarrow y' + \frac{1}{x} y = \frac{1}{x^2} \rightsquigarrow (xy)' = \frac{1}{x}$$

$$\rightsquigarrow xy = \ln(-x) + C$$

$$\rightsquigarrow y = \frac{1}{x}(\ln(-x) + C)$$

$$y(-1) = 2 \rightsquigarrow -C = 2 \rightsquigarrow C = -2$$

$$\boxed{y = \frac{1}{x}(\ln(-x) - 2)}$$

Use Mathematica to visualize the direction field:

$$\boxed{y' = -\frac{1}{x} y + \frac{1}{x^2}}$$

Separation of variables

$$y' = yt^2 \rightsquigarrow \underbrace{\frac{y'}{y} = t^2}_{\text{y and t are separated}} \rightsquigarrow \ln|y| = \frac{t^3}{3} + C \rightsquigarrow \ldots$$

Rewrite: $\frac{dy}{dt} = yt^2 \rightsquigarrow \frac{dy}{y} = t^2 dt \rightsquigarrow$ Integrate both sides

$y(0) = 1$ vs $y(0) = -1$.